UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 7TH FEBRUARY 2013

Organised by the **United Kingdom Mathematics Trust from the School of Mathematics, University of Leeds**

http://www.ukmt.org.uk

SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates. More comprehensive solutions are on the website.

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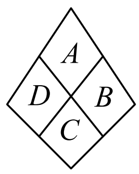
1. **D** In order to be a multiple of 6, a number must be both even and a multiple of 3. Of the numbers given, only B 999 998 and D 999 996 are even. Using the rule for division by 3, we see that, of these two, only 999 996 is a multiple of 3.

2. **B** 180 000 eggs per hour is equivalent to 3000 eggs per minute, i.e. to 50 eggs per second.

3. **E** The figure is itself a quadrilateral. It can be divided into four small quadrilaterals labelled A, B, C, D. There are also four quadrilaterals formed in each case by joining together two of the smaller quadrilaterals: A and B; B and C; C and D; D and A.



4. **D** The number of seeds in a special packet is $1.25 \times 40 = 50$. So the number of seeds which germinate is $0.7 \times 50 = 35$.

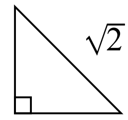
5. **E** A wheatear travels the distance of almost 15 000 km in approximately 50 days. This is on average roughly 300 km per day.

6. **E** In order, the values of the expressions given are: $1 - 0 = 1$; $2 - 1 = 1$; $9 - 8 = 1$; $64 - 81 = -17$; $625 - 1024 = -399$.

7. A Only two colours are needed for the upper four faces of the octahedron. If, for example, blue and red are used then these four faces may be painted alternately red and blue. Consider now the lower four faces: every face adjacent to an upper blue face may be painted red and every face adjacent to an upper red face may be painted blue. So only two colours are required for the whole octahedron.

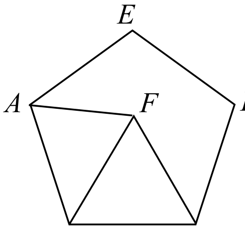
8. D Let the number of scores of 1 be n. Then the product of the scores is $1^n \times 2 \times 3 \times 5 = 30$. Therefore $1 \times n + 2 + 3 + 5 = 30$, i.e. $n = 20$. So Jim threw 23 dice.

9. A Let the length of the shorter sides of the cards be 1 unit. Then, by Pythagoras' Theorem, the length of the hypotenuse of each card is $\sqrt{1^2 + 1^2} = \sqrt{2}$.



So the lengths of the perimeters of the five figures in order are: $4\sqrt{2}$; $4 + 2\sqrt{2}$; $4 + 2\sqrt{2}$; 6; $4 + 2\sqrt{2}$. Also, as $\left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2\frac{1}{4} > 2$ we see that $\frac{3}{2} > \sqrt{2}$. Therefore, $4\sqrt{2} < 6 < 4 + 2\sqrt{2}$. So figure A has the shortest perimeter.

10. C The sum of the interior angles of a pentagon is 540° so $\angle ABC = 540° \div 5 = 108°$. Each interior angle of an equilateral triangle is 60°, so $\angle FBC = 60°$.

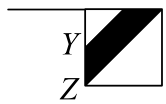


Therefore $\angle ABF = 108° - 60° = 48°$. As $ABCDE$ is a regular pentagon, $BC = AB$. However, $BC = FB$ since triangle BFC is equilateral.
So triangle ABF is isosceles with $FB = AB$.
Therefore $\angle FAB = \angle AFB = (180° - 48°) \div 2 = 66°$.

11. C We first look at $66 = 2 \times 3 \times 11$. Its factors involve none, one, two or all three of these primes. So the factors are 1, 2, 3, 11, 6, 22, 33, 66; and their sum is $144 = 12^2$. Similarly, we can check that the sum of the factors of 3, 22, 40 and 70 is, respectively, $4 = 2^2$, $36 = 6^2$, 90 and $144 = 12^2$. So 40 is the only alternative for which the sum of the factors is not a square number.

12. D As the words 'three' and 'five' contain 5 and 4 letters respectively, their 'sum' will be a 9-letter word. Of the alternatives given, only 'seventeen' contains 9 letters.

13. B The diagram shows the top-right-hand portion of the square. The shaded trapezium is labelled $QXYZ$ and W is the point at which ZY produced meets PQ.



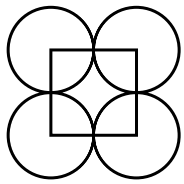
As $QXYZ$ is an isosceles trapezium, $\angle QZY = \angle ZQX = 45°$.
Also, as YX is parallel to ZQ, $\angle XYW = \angle WXY = 45°$. So WYX and WZQ are both isosceles right-angled triangles. As $\angle ZWQ = 90°$ and Z is at the centre of square $PQRS$, we deduce that W is the midpoint of PQ. Hence $WX = XQ = \frac{1}{4}PQ$. So the ratio of the side-lengths of similar triangles WYX and WZQ is 1 : 2 and hence the ratio of their areas is 1 : 4.
Therefore the area of trapezium $QXYZ = \frac{3}{4} \times$ area of triangle $ZWQ = \frac{3}{32} \times$ area $PQRS$ since triangle ZWQ is one-eighth of $PQRS$. So the fraction of the square which is shaded is $4 \times \frac{3}{32} = \frac{3}{8}$.

14. D As all the fractions are raised to the power 3, the expression which has the largest value is that with the largest fraction in the brackets.
Each of these fractions is a little larger than $1\frac{1}{2}$. Subtracting $1\frac{1}{2}$ from each in turn, we get the fractions $\frac{1}{14}, \frac{1}{6}, \frac{1}{4}, \frac{3}{10}, 0$, the largest of which is $\frac{3}{10}$ (because $0 < \frac{1}{14} < \frac{1}{6} < \frac{1}{4} = \frac{2\frac{1}{2}}{10} < \frac{3}{10}$). Hence $\left(\frac{9}{5}\right)^3$ is the largest.

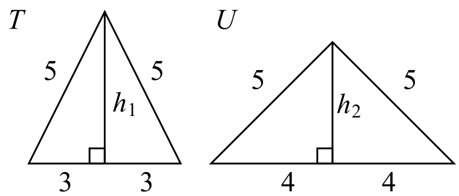
15. B From the information given, we may deduce that the number of coins is a multiple of each of 3, 5, 7. Since these are distinct primes, their lowest common multiple is $3 \times 5 \times 7 = 105$. So the number of coins in the bag is a multiple of 105. So there are 105 coins in the bag since 105 is the only positive multiple of 105 less than or equal to 200.

16. A The image of a straight line under a rotation is also a straight line. The centre of rotation, the point (1, 1), lies on the given line and so also lies on the image. The given line has slope 1 and so its image will have slope -1. Hence graph A shows the image.

17. D The radius of each disc in the figure is equal to half the side-length of the square, i.e. $\frac{1}{\pi}$. Because the corners of a square are right-angled, the square hides exactly one quarter of each disc. So three-quarters of the perimeter of each disc lies on the perimeter of the figure. Therefore the length of the perimeter is $4 \times \frac{3}{4} \times 2\pi \times \frac{1}{\pi} = 6$.

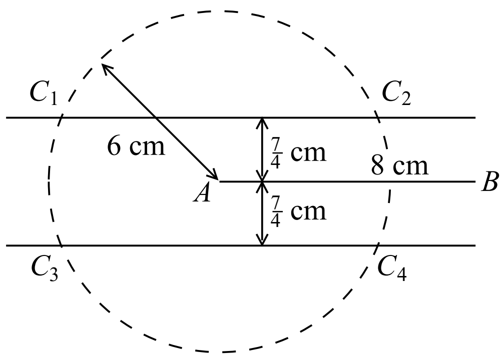


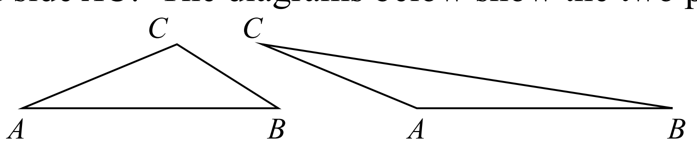
18. C The diagrams show isosceles triangles T and U. The perpendicular from the top vertex to the base divides an isosceles triangle into two congruent right-angled triangles as shown in both T and U. Evidently, by Pythagoras' Theorem, $h_1 = 4$ and $h_2 = 3$. So both triangles T and U consist of two '3, 4, 5' triangles and therefore have equal areas.



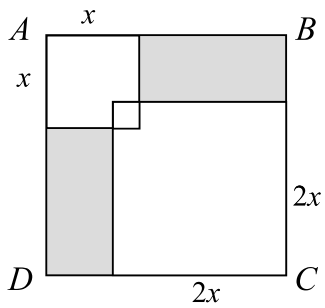
19. E $\left(x \div (y \div z)\right) \div \left((x \div y) \div z\right) = \left(x \div \frac{y}{z}\right) \div \left(\left(\frac{x}{y}\right) \div z\right) = \left(x \times \frac{z}{y}\right) \div \left(\frac{x}{y} \times \frac{1}{z}\right)$
$= \frac{xz}{y} \div \frac{x}{yz} = \frac{xz}{y} \times \frac{yz}{x} = z^2$.

20. B Let the base AB of the triangle be the side of length 8 cm and let AC be the side of length 6 cm. So C must lie on the circle with centre A and radius 6 cm as shown. The area of the triangle is to be 7 cm^2, so the perpendicular from C to AB (or to BA produced) must be of length $\frac{7}{4}$ cm.
The diagram shows the four possible positions of C. However, since $\angle BAC_1 = \angle BAC_3$ and $\angle BAC_2 = \angle BAC_4$, these correspond to exactly two possibilities for the length of the third side AC. The diagrams below show the two possibilities.

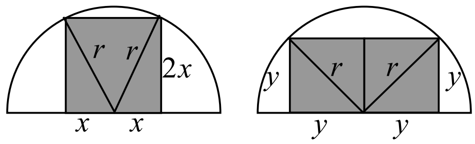




21. B The large square has area $196 = 14^2$. So it has side-length 14. The ratio of the areas of the inner squares is $4 : 1$, so the ratio of their side-lengths is $2 : 1$. Let the side-length of the larger inner square be $2x$, so that of the smaller is x. The figure is symmetric about the diagonal AC and so the overlap of the two inner squares is also a square which therefore has side-length 1. Thus the vertical height can be written as $x + 2x - 1$. Hence $3x - 1 = 14$ and so $x = 5$. Also, the two shaded rectangles both have side-lengths $2x - 1$ and $x - 1$; that is 9 and 4. So the total shaded area is 72.

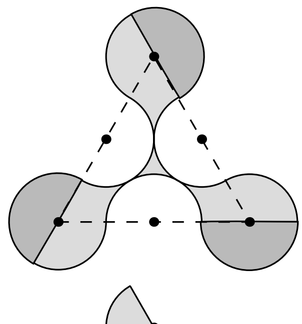


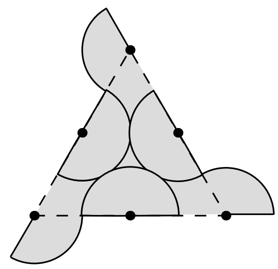
22. D Let the radius of each semicircle be r. In the left-hand diagram, let the side-length of the square be $2x$. By Pythagoras' Theorem, $x^2 + (2x)^2 = r^2$ and so $5x^2 = r^2$. So this shaded area is $4x^2 = \frac{4r^2}{5}$. In the right-hand diagram, let the side-length of each square be y. Then by Pythagoras' Theorem, $y^2 + y^2 = r^2$ and so this shaded area is r^2. Therefore the ratio of the two shaded areas is $\frac{4}{5} : 1 = 4 : 5$.



23. A If Alfred is telling the truth, the other three are lying (as their statements would then be false) and we know this is not the case. Hence Alfred is lying. Similarly, if Horatio is telling the truth, the other three are lying which again cannot be the case. So Horatio is lying. Hence the two who are telling the truth are Bernard and Inigo. (A case where this situation would be realised would be if the brothers in descending order of age were Alfred , Bernard, Horatio and Inigo.)

24. B The length of the side of the triangle is equal to four times the radius of the arcs. So the arcs have radius $2 \div 4 = \frac{1}{2}$. In the first diagram, three semicircles have been shaded dark grey. The second diagram shows how these semicircles may be placed inside the triangle so that the whole triangle is shaded. Therefore the difference between the area of the shaded shape and the area of the triangle is the sum of the areas of three sectors of a circle. The interior angle of an equilateral triangle is 60°, so the angle at the centre of each sector is $180° - 60° = 120°$. Therefore each sector is equal in area to one-third of the area of a circle. Their combined area is equal to the area of a circle of radius $\frac{1}{2}$. So the required area is $\pi \times \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$.





25. D $\left(10^{640} - 1\right)$ is a 640-digit number consisting entirely of nines. So $\frac{\left(10^{640} - 1\right)}{9}$ is a 640-digit number consisting entirely of ones.

Therefore $\frac{10^{641} \times \left(10^{640} - 1\right)}{9}$ consists of 640 ones followed by 641 zeros. So $\frac{10^{641} \times \left(10^{640} - 1\right)}{9} + 1$ consists of 640 ones followed by 640 zeros followed by a single one. Therefore it has 1281 digits.

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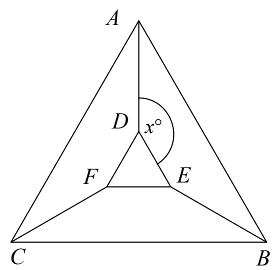
1. **A** 25% of $\frac{3}{4} = \frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$.

2. **D** The first four options are the smallest positive integers which are both odd and not prime. However, the next largest odd numbers after 9, 15, 21 are 11, 17, 23 respectively and these are all prime. The next largest odd number after 25 is 27, which is not prime. So 25 is the smallest positive integer which satisfies all three conditions.

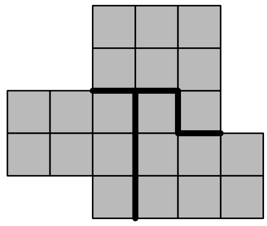
3. **E** Clearly AD lies along one of the lines of symmetry of the figure. So $\angle FDA = \angle EDA = x°$. Triangle DEF is equilateral so $\angle EDF = 60°$.
The angles which meet at a point sum to 360°, so
$x + x + 60 = 360$.
Therefore $x = 150$.



4. **C** Since m is even, $m = 2k$ for some integer k. So $3m + 4n = 2(3k + 2n)$; $5mn = 2(5kn)$; $m^3n^3 = 8k^3n^3$ and $5m + 6n = 2(5k + 3n)$, which are all even. As n is odd, $3n$ is also odd. So $m + 3n$ is an even integer plus an odd integer and is therefore odd. The square of an odd integer is odd so $(m + 3n)^2$ is odd.

5. E In one complete cycle of 4 hours, the clock is struck $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$ times. So in 24 hours the clock is struck $6 \times 36 = 216$ times.

6. E The large shape consists of 21 small squares, so the required shape is made up of 7 small squares. So A and C may be eliminated. The diagram on the right shows that shape E is as required. It is left to the reader to check that neither B nor D was the shape used.

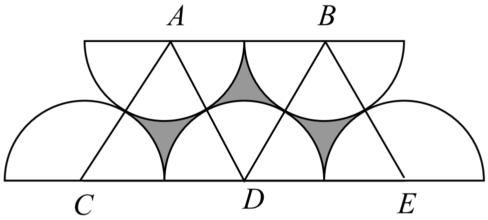


7. B Since 6 and 15 are factors of the integer, its prime factors will include 2, 3 and 5. So 10 and 30 will also be factors of the required integer. Seven of its factors are now known and as 1 must also be a factor, the required integer is 30, the factors of which are 1, 2, 3, 5, 6, 10, 15, 30.
(*Positive integers with exactly 8 factors are of the form pqr or pq^3 or p^7 where p, q, r are distinct primes*.)

8. C The missing die, if correctly placed in the figure, would show faces 1, 3, 5 placed in a clockwise direction around the nearest corner. An examination of each of the five proposed dice shows that only C has this property.

9. A Gill's car uses $p/100$ litres of petrol for every one kilometre travelled. So for a journey of length d km, $pd/100$ litres of petrol are required.

10. B A, B, C, D, E are the centres of the five semicircles. Note that AC joins the centres of two touching semicircles and therefore passes through the point of contact of the semicircles. So AC has length $2 + 2 = 4$. This also applies to all of the other sides of triangles ACD and BED. Hence both triangles are equilateral. So each of the nine arcs which make up the perimeter of the shaded shape subtends an angle of 60° at the centre of a semicircle.



So the length of the perimeter of the shaded figure is $9 \times \frac{1}{6} \times 2 \times \pi \times 2 = 6\pi$.

11. C Precisely one of Jenny and Willie is telling the truth since the number of people is either even or odd. Similarly, precisely one of Sam and Mrs Scrubitt is telling the truth since the number of people is either a prime number or a number which is the product of two integers greater than one. So although it is not possible to deduce who is telling the truth, it is possible to deduce that exactly two of them are doing so.

12. D Let the width of each strip be 1 unit. Then the triangle has base 8 and perpendicular height 8. So its area is equal to $\frac{1}{2} \times 8 \times 8 = 32$. Looking from the right, the area of the first shaded strip is 1 unit of area less than the first unshaded strip. This difference of 1 unit also applies to the other three pairs of strips in the triangle, which means that the shaded area is 4 less than the unshaded area. So the total shaded area is $\frac{1}{2}(32 - 4) = 14$. Therefore the required fraction is $\frac{14}{32} = \frac{7}{16}$.

13. B The smallest such number is $1 + 2 = 3$, whilst the largest is $99 + 100 = 199$. Every number between 3 and 199 may be written as $1 + n$ with $n = 2, 3, \ldots, 99$ or as $100 + n$ with $n = 1, \ldots, 99$. So in total there are $(199 - 3) + 1 = 197$ such numbers.

14. B Chris Froome's average speed $\approx \frac{3400}{84}$ km/h $\approx \frac{3400}{85}$ km/h $= \frac{200}{5}$ km/h $= 40$ km/h.

15. E Let Zac's number be x. Then $\frac{1}{2}x + 8 = 2x - 8$. So $x + 16 = 4x - 16$. Therefore $32 = 3x$, that is $x = 10\frac{2}{3}$.

16. C If the areas of the original and new triangles are the same then the product of the base and the perpendicular height must be the same for the two triangles. When the base of the original triangle is increased by 25%, its value is multiplied by $\frac{5}{4}$. So if the area is to remain unchanged then the perpendicular height must be multiplied by $\frac{4}{5}$, which means that its new value is 80% of its previous value. So it is decreased by 20%.

17. D The number of minutes in one week is $7 \times 24 \times 60$, which may be written as $7 \times (6 \times 4) \times (5 \times 3 \times 2 \times 2) = (7 \times 6 \times 5 \times 4 \times 3 \times 2) \times 2$. So the number of weeks in $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ minutes is $8 \div 2 = 4$.

18. B The point (m, n) is hidden if and only if m and n share a common factor greater than 1. So (6, 2) is hidden by (3, 1) since 6 and 2 have common factor 2. Also (6, 3) is hidden by (2, 1) whilst (6, 4) is hidden by (3,2). However, 6 and 5 have no common factor other than 1 and therefore (6, 5) is not a hidden point.

19. C Note that $8^m = (2^3)^m = 2^{3m} = (2^m)^3$ and $27 = 3^3$; so $2^m = 3$. Therefore $4^m = 2^m \times 2^m = 9$.

20. D Each exterior angle of a regular pentagon is $\frac{1}{5} \times 360° = 72°$. So each of the five circular arcs has radius 2 and so subtends an angle of $(180 + 72)°$ at a vertex of the pentagon. Therefore the area of each of the five shaded major sectors is $\frac{252}{360} \times \pi \times 2^2 = \frac{7}{10} \times \pi \times 4 = \frac{14\pi}{5}$. So the total shaded area is 14π.

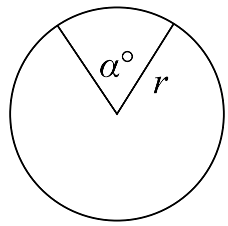
21. D Firstly suppose that any two knights X and Y win x and y bouts respectively and that x is at least as large as y. The difference between their total scores would be the same as if X had won $x - y$ bouts and Y had won none, since each of the separate totals would have been reduced by the same amount, namely $20y$. A similar procedure applies to losses. For example, if X won 3 and lost 6, while Y won 8 and lost 2, the difference between their total scores is the same as if X won 0 and lost 4, while Y won 5 and lost 0. In each case the difference is 32.

This argument shows that, in the case of the Black Knight, B, and the Red Knight, R, the smallest number of bouts will be achieved when one of B, R wins all his bouts and the other loses all his bouts. Also B has to score one more point than R. The possible scores for the knight who wins all his bouts are 20, 40, 60, 80, 100, 120, … . while the possible scores for the knight who loses all his bouts are 17, 34, 51, 68, 85, 102, 119, 136, … .

The first two numbers to differ by 1 are 119 and 120. Thus the Black Knight has a total of 120 corresponding to winning all of his 6 bouts and the Red Knight has a total of 119 corresponding to losing all of his 7 bouts.

22. B Let $a = a_1a_2$ where a_2 is the largest square dividing a. Note that a_1 is then a product of distinct primes. Similarly write $b = b_1b_2$ and $c = c_1c_2$. Since ab is a square, a_1b_1 must be a square; so $a_1 = b_1 = k$ say. Similarly $c_1 = k$. The smallest possible value of k is 2 (since a is not a square); and the smallest possible values for a_2, b_2, c_2 are 1, 4 and 9 in some order. This makes $a + b + c = 2 + 8 + 18 = 28$.

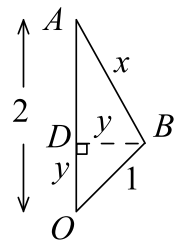
23. A Let the radius of the circle be r and let the angle of the sector be $\alpha°$.



Then the perimeter of the sector is $2r + \frac{\alpha}{360} \times 2\pi r$.
This equals $2\pi r$, the circumference of the original circle.
So $2r + \frac{\alpha}{360} \times 2\pi r = 2\pi r$.
Therefore the fraction of the area of the disc removed is
$\frac{\alpha}{360} = \frac{2\pi r - 2r}{2\pi r} = \frac{\pi - 1}{\pi}$.

24. A There are 9000 four-digit integers. To calculate the number of these which have four different digits, we note that we have a choice of 9 for the thousands digit. We now have a choice of 9 for the hundreds digit (since we can choose 0 as a possible digit). After these two digits have been chosen, we have a choice of 8 for the tens digit and then 7 for the units digit. So the number of four-digit numbers in which all digits are different is $9 \times 9 \times 8 \times 7$.
Therefore the number of four-digit numbers which have at least one digit repeated is $9000 - 9 \times 9 \times 8 \times 7 = 9(1000 - 9 \times 8 \times 7) =$
$9 \times 8 \times (125 - 9 \times 7) = 72 \times (125 - 63) = 72 \times 62$.

25. E Let each side of the octagon have length x. The octagon may be divided into eight triangles by joining the centre of the circle to the vertices of the octagon. One such triangle is shown. Each of these triangles has one side of length 1 (the radius of the smaller circle), one side of length 2 (the radius of the larger circle) and one side of length x. So all eight triangles are congruent. Therefore $\angle AOB = 360° \div 8 = 45°$.



Let D be the foot of the perpendicular from B to AO. Then triangle BDO is an isosceles right-angled triangle.
Let $OD = DB = y$. Applying Pythagoras' Theorem to triangle BDO:
$y^2 + y^2 = 1$. So $y = \frac{1}{\sqrt{2}}$.
Applying Pythagoras' Theorem to triangle ADB:
$x^2 = (2 - y)^2 + y^2 = \left(2 - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 4 - 2 \times 2 \times \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2} = 5 - \frac{4}{\sqrt{2}} = 5 - 2\sqrt{2}$.
So the length of the perimeter is $8x = 8\sqrt{5 - 2\sqrt{2}}$.
(*Note that the length of AB may also be found by applying the Cosine Rule to triangle OAB.*)